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Pythagoras's theorem on a two-dimensional lattice from a 'natural' Dirac operator and Connes's distance formula

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Abstract

One of the key ingredients of Connes's noncommutative geometry is a generalized Dirac operator which induces a metric (Connes's distance) on the pure state space. We generalize such a Dirac operator devised by Dimakis *et al*, whose Connes distance recovers the linear distance on an one-dimensional lattice, to the two-dimensional case. This Dirac operator has the *local eigenvalue property* and induces a Euclidean distance on this two-dimensional lattice, which is referred to as 'natural'. This kind of Dirac operator can be easily generalized into any higher-dimensional lattices.

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Notations. σ_i, $i = 1, 2, 3$ are Pauli matrices defined in the ordinary way. If S is a finite set, then $|S|$ is the number of elements in S. Let $i = 1, 2, 3$, define γ-matrices as

$$\gamma^i = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix} \qquad \gamma^4 = \begin{pmatrix} & \mathrm{i} \\ -\mathrm{i} & \end{pmatrix}.$$

The vacant matrix elements are understood as zeros and this will be taken as a convention throughout this paper. Introduce $\gamma^1_\pm = \frac{1}{2}(\gamma^1 \pm \mathrm{i}\gamma^2)$, $\gamma^2_\pm = \frac{1}{2}(\gamma^3 \pm \mathrm{i}\gamma^4)$, whose explicit matrix representations are taken to be

$$\gamma^1_+ = \begin{pmatrix} & & & 1 \\ & & 0 & \\ & 1 & & \\ 0 & & & \end{pmatrix} \qquad \gamma^1_- = \begin{pmatrix} & & & 0 \\ & & 1 & \\ & 0 & & \\ 1 & & & \end{pmatrix}$$

$$\gamma^2_+ = \begin{pmatrix} & & 0 & \\ & & & -1 \\ 1 & & & \\ & 0 & & \end{pmatrix} \qquad \gamma^2_- = \begin{pmatrix} & & 1 & \\ & & & 0 \\ 0 & & & \\ & -1 & & \end{pmatrix}$$

$\gamma_\pm^i$, $i = 1, 2$ satisfy Clifford algebra relations

$$\{\gamma_\pm^i, \gamma_\pm^j\} = 0 \qquad \{\gamma_\pm^i, \gamma_\mp^j\} = \delta^{ij} \qquad i, j = 1, 2.$$

Convention $a = 1$ for the lattice constant a is adopted in this paper.

1. Introduction

The lattice Dirac operator is a long-standing problem embarrassing lattice field theorists. The no-go theorem [1] makes the implementation of chiral fermions on a lattice almost impossible. In recent years, some breakthroughs have been achieved, e.g. the rediscovery of the Ginsparg–Wilson relation [2], the domain-wall devices and overlap Dirac operators [3, 4]. On the other hand, the lattice can be considered as a simplest realization of noncommutative geometry (NCG), which has drawn more and more attention of theoretical physicists due to its applications in the standard model of particle physics [5–7], lattice field theory [8–10] and string/M-theory [11, 12]. NCG provides a powerful candidate for a mathematical framework for the geometrical understanding of fundamental physical laws. In Connes's version of NCG, a generalized Dirac operator plays a central role in introducing the metric structure onto a noncommutative space [13–15]. Then intuitively, it becomes an interesting question whether Connes's NCG method could elucidate the problem of the lattice Dirac operator. In fact, some groups have explored this question. Bimonte *et al* [16] first pointed out that the naive Dirac operator is not able to induce the conventional distance on a lattice by Connes's construction and the Wilson–Dirac operator gives an even worse result. Starting from this observation, Atzmon [17] computed this *anomalous distance* induced by the naive Dirac operator for a one-dimensional lattice precisely, which gives a quite striking result:

$$d(0, 2n-1) = 2n \qquad d(0, 2n) = 2\sqrt{n(n+1)} \qquad \forall n \in \mathbb{N}.$$

On a finite one-dimensional lattice, Dimakis and Müller-Hoissen [18] discovered a Dirac operator whose Connes distance coincides with the usual linear distance. This series of work indicates that it is a highly nontrivial question to devise a proper Dirac operator whose Connes distance is a desired one. In fact, Iochum *et al* [19] gave a confirmative answer to this question for finite spaces. Therefore, NCG provides a criterion for the rational choices of Dirac operator on lattices. We shall refer to those that restore Euclidean geometry as '*natural*', so naive and Wilson operators are 'unnatural' in this sense.

In this paper, we show that the existence theorem of Iochum *et al* is valid for a two-dimensional infinite lattice, by constructing a 'natural' Dirac operator on this lattice. Thus Pythagoras's theorem holds for Connes's distance. The restoration of metric relies on the fact that this operator has a so-called *local eigenvalue property*, such that the norm $\|[\mathcal{D}, f]\|$ is completely solvable. The Dirac operator devised by us can be regarded as a generalization of Dimakis's operator in the one-dimensional case; it can be easily generalized into any high-dimensional lattices.

The paper is organized as follows. In section 2, we give a brief introduction to Connes's NCG and his distance formula. The 'natural' Dirac operator is defined in section 3. In section 4, we explore the local eigenvalue property of our Dirac operator. In section 5, we prove that our Dirac operator induces Pythagoras's theorem on a two-dimensional lattice. In section 6, we generalize the 'natural' Dirac operator into any higher-dimensional lattices.

2. Noncommutative geometry and Connes's distance formula

The object of a Connes NCG is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, referred to as a K-cycle. $\mathcal{A}$ is an involutive algebra, represented faithfully and unitarily as a subalgebra of $B(\mathcal{H})$, an algebra of bounded operators on a Hilbert space $\mathcal{H}$. $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$, which is called the generalized Dirac operator; $\mathcal{D}$ has compact resolvent so $[\mathcal{D}, \hat{a}]$ is bounded for all a in $\mathcal{A}$. Here we use $\hat{a}$ to denote the image of a in $B(\mathcal{H})$; without introducing any misunderstanding, we will just omit the hat on a below. A K-cycle is required to satisfy several axioms such that it recovers the ordinary spin geometry on a differential manifold when $\mathcal{A}$ is taken to be the algebra of smooth functions over this spin manifold, $\mathcal{H}$ is the space of L^2-spinors and $\mathcal{D}$ is the classical Dirac operator [15].

From a K-cycle, we can define a metric $d_{\mathcal{D}}(,)$ on the pure state space of $\mathcal{A}$ denoted as $S(\mathcal{A})$. This induced metric is computed by Connes's distance formula

$$d_{\mathcal{D}}(\phi, \psi) = \sup\{|\phi(a) - \psi(a)| : a \in \mathcal{A}, \|[\mathcal{D}, a]\| \leqslant 1\} \tag{1}$$

for any $\phi, \psi \in S(\mathcal{A})$. One can check that $d_{\mathcal{D}}(,)$ satisfies all of the three axioms for a metric. If $\mathcal{A}$ is commutative, pure states correspond to characters which can be interpreted as points, with $\mathcal{A}$ being the algebra of functions over the space of these points (Gel'fand–Naimark theorem); in this circumstance, equation (1) can be rewritten as

$$d_{\mathcal{D}}(p, q) = \sup\{|f(p) - f(q)| : f \in \mathcal{A}, \|[\mathcal{D}, f]\| \leqslant 1\} \qquad \forall p, q \in S(\mathcal{A}). \tag{2}$$

We notice the fact that

$$\|a\|^2 = \sup\{(a\psi, a\psi) : \psi \in \mathcal{H}, \|\psi\| = 1\} = \sup\{\lambda(a) : a^*a\psi = \lambda(a)\psi\} \qquad a \in B(\mathcal{H}).$$

The inequality constraint in equation (2) can be expressed by using eigenvalues of $[\mathcal{D}, f]^\dagger[\mathcal{D}, f]$

$$d_{\mathcal{D}}(p, q) = \sup\{|f(p) - f(q)| : f \in \mathcal{A}, \lambda([\mathcal{D}, f]^\dagger[\mathcal{D}, f]) \leqslant 1\} \qquad \forall p, q \in S(\mathcal{A}). \tag{3}$$

3. 'Natural' Dirac operator on a two-dimensional lattice

First we give a detailed reformulation of Dimakis's operator on a one-dimensional lattice. We coordinatize the one-dimensional lattice by the set of integers $\mathbb{Z}$. Let $\mathcal{A}_0$ be the algebra of complex functions on $\mathbb{Z}$, and $\mathcal{H} = l^2(\mathbb{Z}) \otimes \mathbb{C}^2$ on which the inner product is defined in the canonical way. The operator developed by Dimakis *et al* acting on $\mathcal{H}$ can be written in the form

$$\mathcal{D} = \sigma_+\partial^+ + \sigma_-\partial^- = \begin{pmatrix} & \partial^+ \\ \partial^- & \end{pmatrix}$$

where $\sigma_\pm = (\sigma_1 \pm \mathrm{i}\sigma_2)/2$, $\partial^\pm : \mathcal{A} \to \mathcal{A}$ are finite differences defined by $\partial^\pm = T^\pm - \mathbf{1}$, $(T^\pm f)(x) = f(x \pm 1)$, $\forall f \in \mathcal{A}, \forall x \in \mathbb{Z}$. The restriction of $\partial^\pm$ to $l^2(\mathbb{Z})$ satisfies $\partial^{+\dagger} = \partial^-$; accordingly $\mathcal{D}$ is selfadjoint on $\mathcal{H}$. $\mathcal{A}$ in the K-cycle is a subalgebra of $\mathcal{A}_0$ defined by $\mathcal{A} = \{f \in \mathcal{A}_0 : \|[\mathcal{D}, f]\| < \infty\}$.

Local eigenvalue property of $\mathcal{D}$. Note that

$$[\mathcal{D}, f] = \begin{pmatrix} & (\partial^+ f)T^+ \\ (\partial^- f)T^- & \end{pmatrix} = -\begin{pmatrix} & T^+ \cdot (\partial^- f) \\ T^- \cdot (\partial^+ f) & \end{pmatrix}$$

and that $[\mathcal{D}, f]^\dagger = -[\mathcal{D}, \bar{f}]$. Introduce a *f-Hamiltonian* $H(\mathrm{d}f) := [\mathcal{D}, f]^\dagger[\mathcal{D}, f]$. It is easy to check that

$$H(\mathrm{d}f) = \begin{pmatrix} |\partial^+ f|^2 & \\ & |\partial^- f|^2 \end{pmatrix}.$$

Consider the eigenvalue equation on $\mathcal{H}$

$$H(\mathrm{d}f)\psi = \lambda(\mathrm{d}f)\psi \Leftrightarrow \begin{array}{ll} |\partial^+ f|^2(x)\psi_1(x) = \lambda\psi_1(x) & \forall x \in \mathbb{Z} \\ |\partial^- f|^2(x)\psi_2(x) = \lambda\psi_2(x) & \forall x \in \mathbb{Z} \end{array}$$

if $\psi = (\psi^1, \psi^2)^{\mathrm{T}}$. The constraint in equation (3) therefore implies

$$\lambda(x, 1) = |\partial^+ f|^2(x) \leqslant 1 \qquad \forall x \in \mathbb{Z} \tag{4}$$

$$\lambda(x, 2) = |\partial^- f|^2(x) \leqslant 1 \qquad \forall x \in \mathbb{Z}. \tag{5}$$

Observe that each eigenvalue of $H(\mathrm{d}f)$ is just related to one link of $\mathbb{Z}$, to which we refer as the *local eigenvalue*. As a comparison, the eigenvalue equation for the one-dimensional naive Dirac operator $\mathcal{D}_N := (\partial^+ - \partial^-)/2$ is

$$\begin{aligned}&\tfrac{1}{4}(|\partial^+ f|^2(x) + |\partial^- f|^2(x))\psi(x) + \tfrac{1}{4}((\partial^+ \bar{f})(x)(\partial^+ f)(x+1)\psi(x+2) \\ &\quad +(\partial^- \bar{f})(x)(\partial^- f)(x-1)\psi(x-2)) = \lambda\psi(x)\end{aligned}$$

for all x in $\mathbb{Z}$, with $\mathcal{A}_N = \mathcal{A}$, $\mathcal{H}_N = l^2(\mathbb{Z})$, which evidently do not have local eigenvalues.

To induce $d_{\mathcal{D}}(,)$ on $\mathbb{Z}$, let $f \in \mathcal{A}$ be subjected to (4) and (5), so that $|f((x+m) - f(x)| \leqslant m, m = 1, 2, 3, \ldots$; $d_{\mathcal{D}}(x + m, x)$ hence has an upper bound m. Let $f_0(x) = x$, then f_0 saturates this upper bound. Thus, we have proved $d_{\mathcal{D}}(m, n) = |m - n|, \forall m, n \in \mathbb{Z}$.

Now we generalize the above construction to a two-dimensional lattice. Parametrize the two-dimensional lattice by $\mathbb{Z}^2$ and let $\mathcal{A}_0$ be the algebra of complex functions on $\mathbb{Z}^2$, $\mathcal{H} := l^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$, and

$$\mathcal{D} = \sum_{i=1}^{2}\sum_{s=\pm} \gamma_s^i \partial_i^s = \begin{pmatrix} & & \partial_2^- & \partial_1^+ \\ & & \partial_1^- & -\partial_2^+ \\ \partial_2^+ & \partial_1^+ & & \\ \partial_1^- & -\partial_2^- & & \end{pmatrix} \tag{6}$$

where $\partial_i^\pm : \mathcal{A} \to \mathcal{A}, i = 1, 2$ are defined by $\partial_i^\pm = T_i^\pm - \mathbf{1}$, $(T_i^\pm f)(x) = f(x \pm \hat{i})$ with $\hat{i}, i = 1, 2$ being the generators of $\mathbb{Z}^2$. The restrictions of $\partial_i^\pm$ on $l^2(\mathbb{Z}^2)$ obey $\partial_i^{\pm\dagger} = \partial_i^\mp$; accordingly, $\mathcal{D}^\dagger = \mathcal{D}$. $\mathcal{A}$ is defined to be $\{f \in \mathcal{A}_0 : \|[\mathcal{D}, f]\| < \infty\}$. By translation invariance of $\mathcal{D}$, it holds that

$$d_{\mathcal{D}}((m, n), (m', n')) = d_{\mathcal{D}}((0, 0), (m' - m, n' - n))$$

so that to consider $d_{\mathcal{D}}((0, 0), (m, n))$ is enough; moreover,

$$d_{\mathcal{D}}((0, 0), (m, n)) = d_{\mathcal{D}}((0, 0), (-n, m)), d_{\mathcal{D}}((0, 0), (m, n)) = d_{\mathcal{D}}((0, 0), (n, m))$$

due to rotation and reflection invariance, so we just need to consider $m \geqslant n \geqslant 0$.

To end this section, we adopt the method in [16] to prove

$$d_{\mathcal{D}}((0, 0), (i, 0)) = i \qquad i = 1, 2, \ldots. \tag{7}$$

Lemma 1. *Let $f \in \mathcal{A}$, $\|[\mathcal{D}, f]\| \leqslant 1$ and $\tilde{f}(m, n) := f(m, 0)$, then $\|[\mathcal{D}, \tilde{f}]\| \leqslant 1$.*

Proof. By the definition of $\mathcal{D}$ in (6), we have

$$[\mathcal{D}, \tilde{f}]\psi = ((\partial_1^+ \tilde{f})(T_1^+ \psi_4), (\partial_1^- \tilde{f})(T_1^- \psi_3), (\partial_1^+ \tilde{f})(T_1^+ \psi_2), (\partial_1^- \tilde{f})(T_1^- \psi_1))^{\mathrm{T}}.$$

Notice the definition of $\tilde{f}$,

$$\begin{aligned}([\mathcal{D}, \tilde{f}]\psi, [\mathcal{D}, \tilde{f}]\psi) &= \sum_{m,n}\left(|\partial_1^+ \tilde{f}|^2(m, n)\left(\sum_{i=1}^{4} |\psi_i'|^2(m, n)\right)\right) \\ &= \sum_{m}\left(|\partial_1^+ f|^2(m, 0)\left(\sum_{n}\sum_{i=1}^{4} |\psi_i'|^2(m, n)\right)\right) \\ &\Rightarrow \|[\mathcal{D}, \tilde{f}]\| = \sup\{|\partial_1^+ f|(m, 0) : m \in \mathbb{Z}\} < \infty. \end{aligned} \tag{8}$$

Define $\hat{\mathcal{H}} := \{\psi \in \mathcal{H} : \psi_i = 0, i = 1, 2, 3\}$, then

$$\|[\mathcal{D}, f]\|_{\hat{\mathcal{H}}} \leqslant \|[\mathcal{D}, f]\| \leqslant 1 \tag{9}$$

and

$$([\mathcal{D}, f]\psi, [\mathcal{D}, f]\psi)|_{\hat{\mathcal{H}}} = \sum_{m,n}(|\partial_1^+ f|^2(m,n)|\psi_4|^2(m+1,n) + |\partial_2^+ f|^2(m,n)|\psi_4|^2(m,n+1)).$$

For any $\psi \in \mathcal{H}$, we define $\hat{\psi} \in \hat{\mathcal{H}}$ by the facts that $\hat{\psi}_4(m,n) = 0, n \neq 0$ and that $\hat{\psi}_4(m,0)$ satisfy
$|\hat{\psi}_4(m,0)|^2 = \sum_n \sum_{i=1}^4 |\psi_i'|^2(m,n)$, for all m. Therefore,

$$([\mathcal{D}, \tilde{f}]\psi, [\mathcal{D}, \tilde{f}]\psi) = \sum_m |\partial_1^+ f|^2(m,0)|\hat{\psi}_4(m,0)|^2 \leqslant ([\mathcal{D}, f]\hat{\psi}, [\mathcal{D}, f]\hat{\psi})|_{\hat{\mathcal{H}}} \leqslant \|[\mathcal{D}, f]\|_{\hat{\mathcal{H}}}.$$

Notice (9), and we have

$$\|[\mathcal{D}, \tilde{f}]\| \leqslant \|[\mathcal{D}, f]\|_{\hat{\mathcal{H}}} \leqslant \|[\mathcal{D}, f]\| \leqslant 1.$$

□

Following this lemma, we can just consider f with $\partial_2^+ f = 0$ and reach the conclusion that $d_{\mathcal{D}}((0,0),(i,0)) \leqslant i, i = 1, 2, \ldots;\ f_0(m,n) = m$ saturates this upper bound. Hence, equation (7) holds.

4. Local eigenvalues

We have to consider the eigenvalue problem in equation (3), when discussing distance $d_{\mathcal{D}}((0,0),(m,n)), m \geqslant n \geqslant 1$. Fortunately, our 'natural' Dirac operator has the *local eigenvalue property* also. A very detailed calculation is given below.

Let

$$D(f) := [\mathcal{D}, f] = \begin{pmatrix} & & (\partial_2^- f)T_2^- & (\partial_1^+ f)T_1^+ \\ & & (\partial_1^- f)T_1^- & -(\partial_2^+ f)T_2^+ \\ (\partial_2^+ f)T_2^+ & (\partial_1^+ f)T_1^+ & & \\ (\partial_1^- f)T_1^- & -(\partial_2^- f)T_2^- & & \end{pmatrix}.$$

One can check $[\mathcal{D}, f]^\dagger = -D(\bar{f})$. Define f-*Hamiltonian*

$$\begin{aligned} H(\mathrm{d}f) &:= [\mathcal{D}, f]^\dagger[\mathcal{D}, f] = -D(\bar{f})D(f) \\ &= \begin{pmatrix} |\partial_1^+ f|^2 + |\partial_2^- f|^2 & (T_2^-(\partial_2^+\bar{f}\partial_1^+ f) - T_1^+(\partial_1^-\bar{f}\partial_2^- f))T_1^+T_2^- \\ (T_1^-(\partial_1^+\bar{f}\partial_2^+ f) - T_2^+(\partial_2^-\bar{f}\partial_1^- f))T_1^-T_2^+ & |\partial_1^- f|^2 + |\partial_2^+ f|^2 \end{pmatrix} \\ &\oplus \begin{pmatrix} |\partial_1^+ f|^2 + |\partial_2^+ f|^2 & (T_2^+(\partial_2^-\bar{f}\partial_1^+ f) - T_1^+(\partial_1^-\bar{f}\partial_2^+ f))T_1^+T_2^+ \\ (T_1^-(\partial_1^+\bar{f}\partial_2^- f) - T_2^-(\partial_2^+\bar{f}\partial_1^- f))T_1^-T_2^- & |\partial_1^- f|^2 + |\partial_2^- f|^2 \end{pmatrix} \end{aligned}$$

with the eigenvalue equation

$$H(\mathrm{d}f)\psi = \lambda(\mathrm{d}f)\psi. \tag{10}$$

Fortunately, equation (10) can be reduced to a collection of equation sets, with each equation set being related to a fundamental plaque $\{(m,n),(m+1,n),(m,n-1),(m+1,n-1)\} \subset \mathbb{Z}^2$. Label each fundamental plaque with its upper-left corner (m,n), and the equation set corresponding to plaque (m,n) is then read as

$$\begin{pmatrix} \rho_1^2 + \rho_4^2 - \lambda & \overline{\Delta_4}\Delta_3 - \overline{\Delta_1}\Delta_2 \\ \Delta_4\overline{\Delta_3} - \Delta_1\overline{\Delta_2} & \rho_2^2 + \rho_3^2 - \lambda \end{pmatrix} \begin{pmatrix} \psi_1(m,n) \\ \psi_2(m+1,n-1) \end{pmatrix} = 0 \tag{11}$$

$$\begin{pmatrix} \rho_3^2 + \rho_4^2 - \lambda & -\overline{\Delta_4}\Delta_1 + \overline{\Delta_3}\Delta_2 \\ -\Delta_4\overline{\Delta_1} + \Delta_3\overline{\Delta_2} & \rho_1^2 + \rho_2^2 - \lambda \end{pmatrix} \begin{pmatrix} \psi_3(m,n-1) \\ \psi_4(m+1,n) \end{pmatrix} = 0 \tag{12}$$

in which $\Delta_1(m,n) := (\partial_1^+ f)(m,n)$, $\Delta_2(m,n) := (\partial_2^+ f)(m+1,n-1)$, $\Delta_3(m,n) := (\partial_1^+ f)(m,n-1)$, $\Delta_4(m,n) := (\partial_2^+ f)(m,n-1)$ and, without misunderstanding, we have omitted the argument (m,n). Let $\Delta_i := \rho_i \mathrm{e}^{\mathrm{i}\theta_i}$, $i = 1,2,3,4$, $\theta := \theta_1 + \theta_3 - \theta_2 - \theta_4$, and

$$\begin{aligned} &A := \rho_1^2 + \rho_4^2 \qquad B := \rho_2^2 + \rho_3^2 \qquad C := \overline{\Delta_4}\Delta_3 - \overline{\Delta_1}\Delta_2 \\ &A' := \rho_3^2 + \rho_4^2 \qquad B' := \rho_1^2 + \rho_2^2 \qquad C' := -\overline{\Delta_4}\Delta_1 + \overline{\Delta_3}\Delta_2 \end{aligned}$$

then the secular equation for equation (11) is

$$(\lambda - A)(\lambda - B) - C\bar{C} = \lambda^2 - (A+B)\lambda + AB - C\bar{C} = 0$$

in which

$$C\bar{C} = \rho_3^2\rho_4^2 + \rho_1^2\rho_2^2 - 2\rho_1\rho_2\rho_3\rho_4\cos\theta.$$

The solutions are

$$\lambda_\pm = \tfrac{1}{2}\left(A + B \pm \sqrt{(A-B)^2 + 4C\bar{C}}\right). \tag{13}$$

Similarly for equation (12),

$$\lambda'_\pm = \tfrac{1}{2}\left(A' + B' \pm \sqrt{(A'-B')^2 + 4C'\bar{C}'}\right) \tag{14}$$

in which

$$C'\bar{C}' = \rho_1^2\rho_4^2 + \rho_2^2\rho_3^2 - 2\rho_1\rho_2\rho_3\rho_4\cos\theta.$$

Therefore, $\|[\mathcal{D}, f]\| \leqslant 1 \Leftrightarrow \forall(m,n)$,

$$\lambda_+(m,n) \leqslant 1 \qquad \lambda'_+(m,n) \leqslant 1. \tag{15}$$

Equivalently, equations (13), (14) and inequality (15) $\Leftrightarrow$

$$1 + \rho_1^2\rho_3^2 + \rho_2^2\rho_4^2 + 2\rho_1\rho_2\rho_3\rho_4\cos\theta \geqslant \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 \tag{16}$$

$$2 \geqslant \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2. \tag{17}$$

There is another constraint, the *closedness condition*

$$\Delta_1 + \Delta_4 = \Delta_2 + \Delta_3. \tag{18}$$

Hence we obtain (16)–(18) as the specific expressions for $\|[\mathcal{D}, f]\| \leqslant 1$ on $\mathbb{Z}^2$.

To end this section, we give equation (7) a second proof using (15). In fact, just notice

$$\rho_1^2 \leqslant \rho_1^2 + \rho_4^2 \leqslant \lambda_+ \leqslant 1$$

and we have $\rho_i \leqslant 1$, $i = 1,2,3,4$. Consequently, $d_{\mathcal{D}}((0,0),(i,0))$ has the upper bound i.

5. Pythagoras's theorem for $d_{\mathcal{D}}(,)$ on $\mathbb{Z}^2$

Let $m \geqslant n \geqslant 1$ below and we claim the following.

Theorem 1 (Pythagoras's theorem on a two-dimensional lattice).

$$d_{\mathcal{D}}((0,0),(p,q)) = \sqrt{p^2 + q^2} \qquad \forall p \geqslant q \geqslant 1. \tag{19}$$

The proof needs only inequalities (17) and (18) and we will treat $q = 1$ and $q = 2,3,4,\ldots$ separately. An important inequality in mathematics analysis is needed:

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \leqslant n \sum_{i=1}^{n} a_i b_i \tag{20}$$

where the equality holds if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

The general philosophy of the proof can be illustrated as follows. First we define the concept of a *canonical path*, which means a subset of $\mathbb{Z}^2$ generated by one point denoted as *start* and the operations T_1^+, T_2^+. Then it must have an *end* if T_1^+, T_2^+ just act finite times. Second we pick out a rectangle subset $L(p,q)$ of $\mathbb{Z}^2$, which is defined as $\{(m,n) : m = 0, 1, \ldots, p, n = 0, 1, \ldots, q\}$, and define the link set of $L(p,q)$, $B(p,q) := \{(m,n,s) : s = 1, 2; s = 1 : m = 0, 1, \ldots, p-1, n = 0, 1, \ldots, q; s = 2 : m = 0, 1, \ldots, p, n = 0, 1, \ldots, q-1\}$. For any $f|_{L(p,q)}$ defined on $L(p,q)$, there is a function Δf defined on $B(p,q)$ correspondingly, with $\Delta f(m,n,1) = (\partial_1^+ f)(m,n)$, $\Delta f(m,n,2) = (\partial_2^+ f)(m,n)$. There is another set induced by $L(p,q)$, the set of all canonical paths starting at (0, 0) and ending at (p,q), written as $\Gamma(p,q)$. Any element $\gamma \in \Gamma(p,q)$ can be considered as $p+q$ sequential links, namely a subset of $B(p,q)$. As for the first step of the proof, we want to show that for any $\gamma \in \Gamma(p,q)$,

$$\left|\sum_{l\in\gamma} \Delta f(l)\right| \leqslant \sqrt{p^2+q^2}$$

if f is subject to (17), (18). Adopting reduction of absurdity, we suppose there is a γ_0 such that $|\sum_{l\in\gamma_0} \Delta f(l)| > \sqrt{p^2+q^2}$. Then by virtue of the closedness condition (18), $|\sum_{l\in\gamma} \Delta f(l)| > \sqrt{p^2+q^2}$, $\forall \gamma \in \Gamma(p,q)$, which implies

$$\sum_{l\in\gamma} |\Delta f(l)|^2 > \frac{p^2+q^2}{p+q}.$$

Therefore

$$\sum_{\gamma\in\Gamma(p,q)} \left(\sum_{l\in\gamma} \Delta f(l)^2\right) > R(p,q)\frac{p^2+q^2}{p+q} \tag{21}$$

where we introduce $R(p,q) = \sum_{\gamma\in\Gamma(p,q)} 1 = |\Gamma(p,q)|$, the number of all canonical paths starting from (0, 0) and ending at (p,q), which is equal to the binomial coefficient $\binom{p+q}{p}$. We change the summation for paths in $\Gamma(p,q)$ in (21) to the summation in $B(p,q)$,

$$\sum_{l\in B(p,q)} (L(l)\Delta f(l)^2) > R(p,q)\frac{p^2+q^2}{p+q} \tag{22}$$

where $L(l)$ is the weight function on $B(p,q)$ introduced by changing the index of the summations. The geometric interpretation of $L(l)$ is the number of canonical paths in $\Gamma(p,q)$ passing through link l, which is expressed as

$$L(m,n,1) = R(m,n)R(p-m-1, q-n)$$
$$L(m,n,2) = R(m,n)R(p-m, q-n-1).$$

We induce the third set from $L(p,q)$, the set $Q(p,q)$ of all fundamental plaques as subsets in $L(p,q)$, which can be expressed as $\{(m,n) : m = 0, 1, \ldots, p-1, n = 0, 1, \ldots, q-1\}$. Rewrite (17) as

$$\sum_{i=1}^{4} |\Delta f|^2(p,i) \leqslant 2 \tag{23}$$

which can be considered as a constraint on the four links in one fundamental plaque p. Now sum (23) for all plaques in $Q(p,q)$

$$\sum_{p\in Q(p,q)} \sum_{i=1}^{4} |\Delta f|^2(p,i) \leqslant 2|Q(p,q)| = 2pq.$$

Again we change the summation for plaques to the summation for links and introduce another weight function $S(l)$,

$$\sum_{l\in B(p,q)} S(l)|\Delta f|^2(l) \leqslant 2|Q(p,q)| \tag{24}$$

where $S(m,0,1) = S(m,q,1) = S(0,n,2) = S(p,n,2) = 1$, $S(\text{other}) = 2$, namely to links l shared by two plaques $S(l) = 2$, otherwise $S(l) = 1$. Using (20) again and noticing $L(l) \neq 0$, we obtain

$$\begin{aligned}
2|Q(p,q)| &\geqslant \sum_{l\in B(p,q)} S(l)|\Delta f|^2(l) \\
&\geqslant \sum_{l\in B(p,q)} \left(\frac{S(l)}{L(l)}\right)(L(l)|\Delta f|^2(l)) \\
&\geqslant \frac{1}{|B(p,q)|}\sum_{l\in B(p,q)} \left(\frac{S(l)}{L(l)}\right)\sum_{l\in B(p,q)}(L(l)|\Delta f|^2(l)) \\
&\Leftrightarrow \frac{2|Q(p,q)||B(p,q)|}{\sum_{l\in B(p,q)}\frac{S(l)}{L(l)}} \geqslant \sum_{l\in B(p,q)}(L(l)|\Delta f|^2(l))
\end{aligned} \tag{25}$$

where $|B(p,q)| = 2pq + p + q$. The contradiction lies between inequalities (22) and (25), if only

$$\frac{2|Q(p,q)||B(p,q)|}{\sum_{l\in B(p,q)}\frac{S(l)}{L(l)}} \leqslant R(p,q)\frac{p^2+q^2}{p+q}. \tag{26}$$

As we declared, the description above is just a general philosophy of proof; since the operation on $L(p,q)$ is complicated, we will just prove the case $q = 1$ for equation (26) and leave cases $q \geqslant 2$ to a revised $L(p,q)$ construction.

Lemma 2. *For $q = 1$, (26) reads*

$$2p(1+3p) \leqslant (1+p^2)\sum_{l\in B(p,1)}\frac{S(l)}{L(l)}. \tag{27}$$

Proof. Using the definitions of $L(l)$ and $S(l)$, we have $\sum_{l\in B(p,1)}\frac{S(l)}{L(l)} = 2(p+\sum_{n=1}^{p}\frac{1}{n})$. Then (27) reduces to show

$$3p^2 \leqslant p^3 + (1+p)\left(\sum_{n=1}^{p}\frac{1}{n}\right). \tag{28}$$

However, (28) is easily checked by $p = 1$, $p = 2$ and $p = 3, 4, \ldots$ respectively. □

Accordingly, we reach the conclusion that $d_{\mathcal{D}}((0,0),(p,1))$, $p = 1, 2, 3, \ldots$, has an upper bound $\sqrt{1+p^2}$.

We define $\tilde{L}(p,q)$ to be a 'folding ruler', $\{(m,0),(m,1),(p-1,n),(p,n) : m = 0,1,\ldots,p, n = 2,\ldots,q\}$, for $p \geqslant q \geqslant 2$, and apply our general philosophy of proof to $\tilde{L}(p,q)$. Namely, we induce $\tilde{B}(p,q)$, $\tilde{P}(p,q)$, $\tilde{Q}(p,q)$ from $\tilde{L}(p,q)$, and introduce weight functions $\tilde{L}(l)$, $\tilde{S}(l)$ on $\tilde{B}(p,q)$.

Lemma 3.

$$\frac{2|\tilde{Q}(p,q)||\tilde{B}(p,q)|}{\sum_{l\in\tilde{B}(p,q)}\frac{\tilde{S}(l)}{\tilde{L}(l)}} \leqslant \tilde{R}(p,q)\frac{p^2+q^2}{p+q}. \tag{29}$$

Proof. $|\tilde{Q}(p,q)| = p+q-1$, $|\tilde{B}(p,q)| = 3p+3q-2$, $\tilde{R}(p,q) = 1+pq$; $\tilde{S}(m,0,2) = \tilde{S}(p-1,n,1) = 2, m = 1,2,\ldots,p-1, n = 1,\ldots,q-1$, $\tilde{S}(\text{other}) = 1$; $\tilde{L}(m,0,1) = 1+q(p-m-1)$, $\tilde{L}(m,0,2) = q, m = 0,1,\ldots,p-1$; $\tilde{L}(p,0,2) = 1$; $\tilde{L}(m,1,1) = q(m+1), m = 0,1,\ldots,p-2$; $\tilde{L}(p-1,n,2) = p(q-n)$, $\tilde{L}(p,n,2) = 1+pn, n = 1,\ldots,q-1$; $\tilde{L}(p-1,n,1) = p, n = 1,\ldots,q$. Thus,

$$\sum_{l\in\tilde{B}(p,q)}\frac{\tilde{S}(l)}{\tilde{L}(l)} = 2+2\left(\frac{p}{q}+\frac{q}{p}\right)+\delta \geqslant 6$$

in which

$$\delta = \frac{1}{q}\left(\sum_{n=1}^{p-1}\frac{1}{n}-1\right)+\frac{1}{p}\left(\sum_{n=1}^{q-1}\frac{1}{n}-1\right)+\sum_{n=1}^{p-1}\frac{1}{1+nq}+\sum_{n=1}^{q-1}\frac{1}{1+np} \geqslant 0.$$

We claim that lemma 3 is implied by

$$\begin{aligned}&(p+q)|\tilde{Q}(p,q)||\tilde{B}(p,q)| \leqslant 3\tilde{R}(p,q)(p^2+q^2)\\ &\qquad\Leftrightarrow (p+q)(p+q-1)(3(p+q)-2) \leqslant 3(1+pq)(p^2+q^2) \qquad \forall p \geqslant q \geqslant 2.\end{aligned} \tag{30}$$

We prove (30) by induction. First, fixing one $q = 2,3,\ldots$, we show that (30) holds when $p = q$. In this case, (30) is reduced to

$$12q^3+2q \leqslant 3q^4+13q^2 \qquad \forall q \geqslant 2.$$

The above inequality can be checked to be valid when $q = 2$, $q = 2$ and $q \geqslant 3$. Second, with this fixed q, we suppose that the statement is valid in the case $p = p_0$, and show that it will also hold for $p = p_0+1$. Without misunderstanding, we drop the subscript '0'. It is sufficient to check

$$9pq+9q^2 \leqslant 9(q-1)p^2+3q^3+7p+4q+3.$$

Since $9pq \leqslant 9(q-1)p^2, \forall p \geqslant q \geqslant 2$, we just check

$$9q^2 \leqslant 3q^3+7p+4q+3$$

which is valid when $q = 2$ and $q \geqslant 3$. □

Accordingly, $d_{\mathcal{D}}((0,0),(p,q))$ has an upper bound $\sqrt{p^2+q^2}$ for all $p \geqslant q \geqslant 2$. Choose $f_{(p,q)}(m,n) = \frac{pm+qn}{\sqrt{p^2+q^2}}$, which can be checked easily to saturate this upper bound. Therefore, theorem 1 follows.

6. Higher-dimension cases

Our 'natural' Dirac operator is able to be generalized easily into any dimension greater than two. Let Γ^i, $i = 1,2,\ldots,2d$ be the generators of $\mathrm{Cl}(E^{2d})$ which satisfy Clifford relations

$$\{\Gamma^i,\Gamma^j\} = 2\delta^{ij} \qquad i,j = 1,2,\ldots,2d. \tag{31}$$

Define $\Gamma^k_\pm = (\Gamma^{2k-1} \pm \mathrm{i}\Gamma^{2k})/2$, $k = 1,2,\ldots,d$, and equation (31) changes form into

$$\{\Gamma^m_\pm,\Gamma^n_\pm\} = 0 \qquad \{\Gamma^m_\pm,\Gamma^n_\mp\} = \delta^{mn} \qquad m,n = 1,2,\ldots,d.$$

Now let

$$\mathcal{D}_{(d)} = \sum_{k=1}^{d} \sum_{s=\pm} \Gamma_s^k \partial_k^s$$

and one can check the *square-root property*

$$(\mathcal{D}_{(d)})^2 = \sum_{k=1}^{d} \partial_k^+ \partial_k^- .$$

Let $\mathcal{H} = l^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2^d}$, then $\mathcal{D}_{(d)}$ is selfadjoint. The work showing that $\mathcal{D}_{(d)}$ is 'natural' for a d-dimensional lattice is in progress.

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